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A bound for functions defined on a normed space

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Abstract
Let X be a complex normed space and f a complex valued function defined on X. Assume that $f(A+\lambda B)(\lambda \in \mathbb{C})$ is an entire function for all $A, B \in X$ and there is a scalar monotone non-decreasing function G defined on $[0, \infty)$, such that $|f(A)| \leqslant G(\|A\|_X)(A \in X)$. It is proved that

$$|f(A) - f(B)| \leqslant \|A - B\|_X G\big(1 + \tfrac{1}{2}(\|A + B\|_X + \|A - B\|_X)\big).$$

Applications of this inequality to perturbations of the regularized determinants of the von Neumann–Schatten operators are also discussed.

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1. The main result

Let X be a complex normed space with a norm $\|\cdot\|_X$ and f a complex valued function defined on X. Assume that $f(A+\lambda B)(\lambda \in \mathbb{C})$ is an entire function for all $A, B \in X$ and there is a scalar monotone non-decreasing function G defined on $[0, \infty)$, such that

$$|f(A)| \leqslant G(\|A\|_X)(A \in X). \tag{1.1}$$

In the paper [8] the inequality

$$|f(A) - f(B)| \leqslant \|A - B\|_X G(1 + \|A\|_X + \|B\|_X) \qquad (A, B \in X) \tag{1.2}$$

was established. It is very useful for various applications, in particular, in the theory of the von Neumann–Schatten operators, Ruston–Grotendieck algebras, Fredholm determinants and in the theory of oscillations; cf [4, 6] and references therein.

In this paper we make inequality (1.2) sharper and consider applications of our inequality to perturbations of the determinants of the von Neumann–Schatten operators.

Theorem 1.1. *Let f be a complex valued function defined on X, such that the function $f(A+\lambda B)(\lambda \in \mathbb{C})$ is entire for all $A, B \in X$. In addition, let there be a monotone non-decreasing function $G : [0,\infty) \to [0,\infty)$, such that (1.1) holds. Then*

$$|f(A) - f(B)| \leqslant \|A - B\|_X G\big(1 + \tfrac{1}{2}\|A+B\|_X + \tfrac{1}{2}\|A-B\|_X\big) \qquad (A, B \in X).$$

Proof. Put

$$g(\lambda) = f\big(\tfrac{1}{2}(A+B) + \lambda(A-B)\big).$$

Then $g(\lambda)$ is an entire function and thanks to the residue theorem,

$$f(A) - f(B) = g(1/2) - g(-1/2) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1/2+r} \frac{g(z)\,\mathrm{d}z}{(z-1/2)(z+1/2)} \qquad (r > 0).$$

So

$$|g(1/2) - g(-1/2)| \leqslant (1/2 + r) \sup_{|z|=1/2+r} \frac{|g(z)|}{|z^2 - 1/4|}.$$

But

$$|z^2 - 1/4| = |(r+1/2)^2 \mathrm{e}^{\mathrm{i}2t} - 1/4| \geqslant (r+1/2)^2 - 1/4 = r^2 + r$$
$$\big(z = \big(\tfrac{1}{2} + r\big)\mathrm{e}^{\mathrm{i}t}, 0 \leqslant t < 2\pi\big).$$

In addition,

$$|g(z)| = \big|f\big(\tfrac{1}{2}(A+B) + \lambda(A-B)\big)\big| = \big|f\big(\tfrac{1}{2}(A+B) + (r+1/2)\,\mathrm{e}^{\mathrm{i}t}(A-B)\big)\big|$$
$$\leqslant G\big(\tfrac{1}{2}\|A+B\|_X + \big(\tfrac{1}{2} + r\big)\|A-B\|_X\big) \quad (|z| = 1/2 + r).$$

Therefore

$$|f(A) - f(B)| = |g(1/2) - g(-1/2)| \leqslant \tfrac{1/2+r}{r^2+r} G\big(\tfrac{1}{2}\|A+B\|_X$$
$$+ \big(\tfrac{1}{2} + r\big)\|A-B\|_X\big) \leqslant \tfrac{1}{r} G\big(\tfrac{1}{2}\|A+B\|_X + \big(\tfrac{1}{2} + r\big)\|A-B\|_X\big).$$

Taking $r = \|A-B\|_X^{-1}$, we get the required result. □

Clearly, theorem 1.1 improves (1.2). It supplements the very interesting recent perturbation results; cf [1, 2, 7, 9] and references therein.

2. Perturbations of determinants

In this section, we apply theorem 1.1 to perturbations of determinants. Everywhere below A and B are linear operators in a separable Hilbert space H. For an integer $p \geqslant 1$, let S_p be the Neumann–Schatten ideal of compact operators in H having the finite norm $N_p(A) = [\mathrm{Trace}(AA^*)^{p/2}]^{1/p}$ where A^* is the adjoint. By $\lambda_j(A)$ $(j = 1, 2, \ldots)$ we denote the eigenvalues of A taking with their multiplicities and arranged in the decreasing order: $|\lambda_j(A)| \geqslant |\lambda_{j+1}(A)|$.

First let $H = \mathbb{C}^n$ be the complex n-dimensional Euclidean space.

Corollary 2.1. *Let A and B be linear operators in $\mathbb{C}^n$. Then for any integer $p \geqslant 1$,*

$$|\det(A) - \det(B)| \leqslant \frac{N_p(A-B)}{n^{n/p}} \left(1 + \frac{1}{2}(N_p(A+B) + N_p(A-B))\right)^n.$$

Indeed, due to the inequality for the arithmetic and geometric mean values,

$$|\det(A)| = \prod_{k=1}^{n} |\lambda_k(A)| \leqslant \left(\frac{1}{n}\sum_{k=1}^{n} |\lambda_k(A)|^p\right)^{n/p} \leqslant \frac{1}{n^{n/p}} N_p^n(A).$$

Now the required result is due to theorem 1.1.

Furthermore, let $A \in S_1$ and I be the unit operator. Then

$$|\det(I - A)| = \left|\prod_{j=1}^{\infty}(1 - \lambda_j(A))\right| \leqslant \prod_{j=1}^{\infty} \mathrm{e}^{|\lambda_j(A)|} \leqslant \mathrm{e}^{N_1(A)}.$$

Now theorem 1.1 implies

Corollary 2.2. *Let $A, B \in S_1$. Then*

$$|\det(I - A) - \det(I - B)| \leqslant eN_1(A - B)\exp\left[\tfrac{1}{2}(N_1(A + B) + N_1(A - B))\right].$$

Recall that for an $A \in S_p (p \geqslant 2)$ the regularized determinant is defined as

$$\det_p(A) := \prod_{j=1}^{\infty}(1 - \lambda_j(A)) \exp\left[\sum_{m=1}^{p-1} \frac{\lambda_j^m(A)}{m}\right].$$

The following inequality is valid:

$$\det_p(A) \leqslant \exp\left[c_p N_p^p(A)\right] \tag{2.1}$$

where constant c_p depends on p only, cf [3, p 1106], [4, p 194]. Besides, $c_2 \leqslant 1/2$, cf [5, Section IV.2]. Below we prove that

$$c_p \leqslant \frac{1}{p\left(1 - \sqrt[p-2]{\frac{p}{p+1}}\right)} \qquad (p > 2). \tag{2.2}$$

Now theorem 1.1 implies

Corollary 2.3. *Let $A, B \in S_p$, $p \geqslant 2$. Then*

$$|\det_p(A) - \det_p(B)| \leqslant N_p(A - B)\exp\left[c_p\left(1 + \tfrac{1}{2}(N_p(A + B) + N_p(A - B))\right)^p\right].$$

Let us prove inequality (2.2). To this end consider the function

$$f(z) := \mathrm{Re}\left(\ln(1 - z) + \sum_{m=1}^{p-1} \frac{z^m}{m}\right) \qquad (z \in \mathbb{C}).$$

For $r \equiv |z| < 1$,

$$|f(z)| = \left|\sum_{m=p}^{\infty} \frac{z^m}{m}\right| \leqslant \sum_{m=p}^{\infty} \frac{r^m}{m} = \int_0^r \sum_{m=p}^{\infty} s^{m-1}\,\mathrm{d}s = \int_0^r s^{p-1} \sum_{k=0}^{\infty} s^k\,\mathrm{d}s = \int_0^r \frac{s^{p-1}\,\mathrm{d}s}{1 - s}.$$

Hence for any $w \in (0, 1)$,

$$|f(z)| \leqslant \frac{r^p}{p(1 - w)} \qquad (r < w). \tag{2.3}$$

Furthermore, take into account that

$$|(1 - z)\,\mathrm{e}^z|^2 = (1 - 2\,\mathrm{Re}\,z + r^2)\,\mathrm{e}^{2\mathrm{Re}\,z} \leqslant \mathrm{e}^{-2\mathrm{Re}\,z + r^2}\,\mathrm{e}^{2\mathrm{Re}\,z} = \mathrm{e}^{r^2} \qquad (z \in \mathbb{C}),$$

since $1 + x \leqslant \mathrm{e}^x$, $x \in \mathbb{R}$. So

$$\left|(1 - z)\exp\left[\sum_{m=1}^{p-1} \frac{z^m}{m}\right]\right| = |(1 - z)\,\mathrm{e}^z| \left|\exp\left[\sum_{m=2}^{p-1} \frac{z^m}{m}\right]\right| \leqslant \exp\left[r^2 + \sum_{m=3}^{p-1} \frac{r^m}{m}\right] \qquad (p > 2).$$

Therefore,

$$f(z) \leqslant r^2 + \sum_{m=3}^{p-1} \frac{r^m}{m} \qquad (z \in \mathbb{C}).$$

But for any $w \in (0, 1)$,

$$\left[r^2 + \sum_{m=2}^{p-1} \frac{r^m}{m}\right] r^{-p} \leqslant h_p(w) \quad (r \geqslant w) \qquad \text{where}$$

$$h_p(w) = w^{-p}\left[w^2 + \sum_{m=3}^{p-1} \frac{w^m}{m}\right] \qquad (p > 2).$$

Thus $f(z) \leqslant h_p(w) r^p$ $(r \geqslant w)$. This inequality and (2.3) imply

$$f(z) \leqslant q_p r^p \qquad (z \in \mathbb{C}, \quad p > 2) \qquad \text{where}$$

$$q_p := \min_{w\in(0,1)} \max\left\{h_p(w), \frac{1}{p(1-w)}\right\}.$$

But the function $h_p(w)$ decreases in $w \in (0, 1)$ and $\frac{1}{p(1-w)}$ increases. So the minimum is attained when $h_p(w) = \frac{1}{p(1-w)}$. This equation is equivalent to the equation

$$x^{p-2} = p(1-x)\left[1 + \sum_{m=1}^{p-3} \frac{x^m}{m+2}\right] \qquad (p > 2). \tag{2.4}$$

To check that this equation has a unique positive root $x_0 < 1$, rewrite it as

$$g(x) := \frac{x^{p-2}}{p(1-x)} - \left(1 + \sum_{m=3}^{p-1} \frac{x^{m-2}}{m}\right) = 0.$$

Clearly, $g(0) = -1$, $g(x) \to +\infty$ as $x \to 1 - 0$. So (2.4) has at least one root from $(0, 1)$. But from (2.4) it follows that a root from $(1, \infty)$ is impossible. Moreover, (2.4) is equivalent to the equation

$$\frac{1}{p(1-x)} = \frac{1}{x^{p-2}} + \sum_{m=3}^{p-1} \frac{x^{m-p}}{m}.$$

The left-hand side of this equation increases and the right-hand part decreases on $(0, 1)$. So the positive root is unique. Thus $q_p = 1/p(1-x_0)$. Furthermore, from (2.4) it follows

$$x_0^{p-2} \leqslant p(1-x_0) \sum_{m=0}^{p-3} x_0^m = p\left(1 - x_0^{p-2}\right)$$

since

$$\sum_{m=0}^{p-3} x_0^m = \frac{1 - x_0^{p-2}}{1 - x_0}.$$

This implies that

$$x_0 \leqslant \sqrt[p-2]{\frac{p}{p+1}} \qquad \text{and thus} \qquad q_p \leqslant \frac{1}{p\left(1 - \sqrt[p-2]{\frac{p}{p+1}}\right)}.$$

But thanks to (2.3),

$$\det_p(A) \leqslant \prod_{j=1}^{\infty} \mathrm{e}^{q_p|\lambda_j(A)|^p} = \exp\left[q_p \sum_{k=1}^{\infty} |\lambda_j(A)|^p\right] \leqslant \exp\left[q_p N_p^p(A)\right].$$

This proves (2.2).

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